LOCAL WELL-POSEDNESS OF NONLINEAR DISPERSIVE EQUATIONS ON MODULATION SPACES

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ABSTRACT. By using tools of time-frequency analysis, we obtain some improved local well-posedness results for the NLS, NLW and NLKG equations with Cauchy data in modulation spaces $\mathcal{M}_{0,s}^{p,1}$.

1. INTRODUCTION AND STATEMENT OF RESULTS

The theory of nonlinear dispersive equations (local and global existence, regularity, scattering theory) is vast and has been studied extensively by many authors. Almost exclusively, the techniques developed so far restrict to Cauchy problems with initial data in a Sobolev space, mainly because of the crucial role played by the Fourier transform in the analysis of partial differential operators. For a sample of results and a nice introduction to the field, we refer the reader to Tao's monograph [12] and the references therein.

In this note, we focus on the Cauchy problem for the nonlinear Schrödinger equation (NLS), the nonlinear wave equation (NLW), and the nonlinear Klein-Gordon equation (NLKG) in the realm of modulation spaces. Generally speaking, a Cauchy data in a modulation space is rougher than any given one in a fractional Bessel potential space and this low-regularity is desirable in many situations. Modulation spaces were introduced by Feichtinger in the 80s [6] and have asserted themselves lately as the "right" spaces in time-frequency analysis. Furthermore, they provide an excellent substitute in estimates that are known to fail on Lebesgue spaces. This is not entirely surprising, if we consider their analogy with Besov spaces, since modulation spaces arise essentially replacing dilation by modulation.

The equations that we will investigate are:

$$(1)\qquad (NLS)\ \ i\frac{\partial u}{\partial t} + \Delta_x u + f(u) = 0,\ u(x,0) = u_0(x),$$

$$(2)\qquad (NLW)\ \ \frac{\partial^2 u}{\partial t^2} - \Delta_x u + f(u) = 0,\ u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

$$(3)\qquad (NLKG)\ \ \frac{\partial^2 u}{\partial t^2} + (I - \Delta_x)u + f(u) = 0,\ u(x,0) = u_0(x),\ \frac{\partial u}{\partial t}(x,0) = u_1(x),$$

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where $u(x,t)$ is a complex valued function on $\mathbb{R}^d\times\mathbb{R}$, $f(u)$ (the nonlinearity) is some scalar function of u, and u_0, u_1 are complex valued functions on $\mathbb{R}^d$.

The nonlinearities considered in this paper will be either power-like

$$p_k(u) = \lambda|u|^{2k}u, k\in\mathbb{N}, \lambda\in\mathbb{R}, \tag{4}$$

or exponential-like

$$e_\rho(u) = \lambda(e^{\rho|u|^2}-1)u, \lambda,\rho\in\mathbb{R}. \tag{5}$$

Both nonlinearities considered have the advantage of being smooth. The corresponding equations having power-like nonlinearities p_k are sometimes referred to as algebraic nonlinear (Schrödinger, wave, Klein-Gordon) equations. The sign of the coefficient λ determines the defocusing, absent, or focusing character of the nonlinearity, but, as we shall see, this character will play no role in our analysis on modulation spaces.

The classical definition of (weighted) modulation spaces that will be used throughout this work is based on the notion of short-time Fourier transform (STFT). For $z=(x,\omega)\in\mathbb{R}^{2d}$, we let M_ω and T_x denote the operators of modulation and translation, and $\pi(z)=M_\omega T_x$ the general time-frequency shift. Then, the STFT of f with respect to a window g is

$$V_gf(z) = \langle f, \pi(z)g\rangle.$$

Modulation spaces provide an effective way to measure the time-frequency concentration of a distribution through size and integrability conditions on its STFT. For $s,t\in\mathbb{R}$ and $1\le p,q\le\infty$, we define the weighted modulation space $\mathcal{M}^{p,q}_{t,s}(\mathbb{R}^d)$ to be the Banach space of all tempered distributions f such that, for a nonzero smooth rapidly decreasing function $g\in\mathcal{S}(\mathbb{R}^d)$, we have

$$\|f\|_{\mathcal{M}^{p,q}_{t,s}} = \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|V_gf(x,\omega)|^p <x>^{tp}\,dx\right)^{q/p} <\omega>^{qs}\,d\omega\right)^{1/q} < \infty.$$

Here, we use the notation

$$<x> = (1+|x|^2)^{1/2}.$$

This definition is independent of the choice of the window, in the sense that different window functions yield equivalent modulation-space norms. When both $s=t=0$, we will simply write $\mathcal{M}^{p,q}=\mathcal{M}^{p,q}_{0,0}$. It is well-known that the dual of a modulation space is also a modulation space, $(\mathcal{M}^{p,q}_{s,t})' = \mathcal{M}^{p',q'}_{-s,-t}$, where p', q' denote the dual exponents of p and q, respectively. The definition above can be appropriately extended to exponents $0<p,q\le\infty$ as in the works of Kobayashi [9], [10]. More specifically, let $\beta>0$ and $\chi\in\mathcal{S}$ such that supp$\hat{\chi}\subset\{|\xi|\le 1\}$ and $\sum_{k\in\mathbb{Z}^d}\hat{\chi}(\xi-\beta k)=1$, $\forall\xi\in\mathbb{R}^d$. For $0<p,q\le\infty$ and $s>0$, the modulation space $\mathcal{M}^{p,q}_{0,s}$ is the set of all tempered distributions f such that

$$\left(\sum_{k\in\mathbb{Z}^d}\left(\int_{\mathbb{R}^d}|f*(M_{\beta k}\chi)(x)|^p\,dx\right)^{\frac{q}{p}} <\beta k>^{sq}\right)^{\frac{1}{q}} < \infty. \tag{6}$$

When, $1 \leq p, q \leq \infty$ this is an equivalent norm on $\mathcal{M}^{p,q}_{0,s}$, but when $0 < p, q < 1$ this is just a quasi-norm. We refer to [9] for more details. For another definition of the modulation spaces for all $0 < p, q \leq \infty$ we refer to [5, 15]. For a discussion of the cases when p and/or $q = 0$, see [4]. These extensions of modulation spaces have recently been rediscovered and many of their known properties reproved via different methods by Baoxiang et all [1], [2]. There exist several embedding results between Lebesgue, Sobolev, or Besov spaces and modulation spaces, see for example [11], [13]; also [1], [2]. We note, in particular, that the Sobolev space H^2_s coincides with $\mathcal{M}^{2,2}_{0,s}$. For further properties and uses of modulation spaces, the interested reader is referred to Gröchenig's book [8].

The goal of this note is two fold: *to improve* some recent results of Baoxiang, Lifeng and Boling [1] on the local well-posedness of nonlinear equations stated above, by allowing the Cauchy data to lie in any modulation space $\mathcal{M}^{p,1}_{0,s}$, $p > \frac{d}{d+1}$, $s \geq 0$, and *to simplify* the methods of proof by employing well-established tools from time-frequency analysis. Ideally, one would like to adapt these methods to deal with global well-posedness as well. We plan to address these issues in a future work.

In what follows, we assume that $d \geq 1, k \in \mathbb{N}, \frac{d}{d+1} < p \leq \infty$, $\lambda, \rho \in \mathbb{R}$ and $s \geq 0$ are given. With p_k and e_ρ defined by (4) and (5) respectively, our main results are the following.

Theorem 1. *Assume that* $u_0 \in \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d)$ *and* $f \in \{p_k, e_\rho\}$. *Then, there exists* $T^* = T^*(\|u_0\|_{\mathcal{M}^{p,1}_{0,s}})$ *such that* (1) *has a unique solution* $u \in C([0, T^*], \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d))$. *Moreover, if* $T^* < \infty$, *then* $\limsup_{t \to T^*} \|u(\cdot, t)\|_{\mathcal{M}^{p,1}_{0,s}} = \infty$.

Theorem 2. *Assume that* $u_0, u_1 \in \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d)$ *and* $f \in \{p_k, e_\rho\}$. *Then, there exists* $T^* = T^*(\|u_0\|_{\mathcal{M}^{p,1}_{0,s}}, \|u_1\|_{\mathcal{M}^{p,1}_{0,s}})$ *such that* (2) *has a unique solution* $u \in C([0, T^*], \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d))$. *Moreover, if* $T^* < \infty$, *then* $\limsup_{t \to T^*} \|u(\cdot, t)\|_{\mathcal{M}^{p,1}_{0,s}} = \infty$.

Theorem 3. *Assume that* $u_0, u_1 \in \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d)$ *and* $f \in \{p_k, e_\rho\}$. *Then, there exists* $T^* = T^*(\|u_0\|_{\mathcal{M}^{p,1}_{0,s}}, \|u_1\|_{\mathcal{M}^{p,1}_{0,s}})$ *such that* (3) *has a unique solution* $u \in C([0, T^*], \mathcal{M}^{p,1}_{0,s}(\mathbb{R}^d))$. *Moreover, if* $T^* < \infty$, *then* $\limsup_{t \to T^*} \|u(\cdot, t)\|_{\mathcal{M}^{p,1}_{0,s}} = \infty$.

Remark 1. In Theorem 1 we can replace the (NLS) equation with the following more general (NLS) type equation:

$$(7) \quad (NLS)_\alpha \;\; i\frac{\partial u}{\partial t} + \Delta_x^{\alpha/2} u + f(u) = 0,\; u(x, 0) = u_0(x),$$

for any $\alpha \in [0, 2]$ and $p \geq 1$. The operator $\Delta_x^{\alpha/2}$ is interpreted as a Fourier multiplier operator (with t fixed), $\widehat{\Delta}_x^{\alpha/2} u(\xi, t) = |\xi|^\alpha \widehat{u}(\xi, t)$. This strengthening will become evident from the preliminary Lemma 1 of the next section.

Remark 2. Theorems 1.1 and 1.2 of [1] are particular cases of Theorem 1 with $p = 2$ and $s = 0$.

2. Fourier multipliers and multilinear estimates

The generic scheme in the local existence theory is to establish linear and nonlinear estimates on appropriate spaces that contain the solution u. As indicated by the main theorems above, the spaces we consider here are $\mathcal{M}^{p,1}_{0,s}$, and we present the appropriate estimates in the lemmas below. In fact, we will need estimates on Fourier multipliers on modulation spaces. As proved in [3] and [7], a function $\sigma(\xi)$ is a symbol of a bounded Fourier multiplier on $\mathcal{M}^{p,q}$ for $1 \leq p, q \leq \infty$ if $\sigma \in W(\mathcal{F}L^1, \ell^\infty)$ (see the proofs of the following two lemmas for a definition of this space). As we shall indicate below, this condition can be naturally extended to give a sufficient criterion for the boundedness of the Fourier multiplier operator on $\mathcal{M}^{p,q}_{0,s}$ for $0 < p, q \leq \infty$ and $s \geq 0$. The notation $A \lesssim B$ stands for $A \leq cB$ for some positive constant c independent of A and B.

Lemma 1. *Let σ be a function defined on $\mathbb{R}^d$ and consider the Fourier multiplier operator H_σ defined by*

$$H_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(\xi)\, \hat{f}(\xi)\, e^{2\pi\xi\cdot x}\, d\xi.$$

Let $\chi \in \mathcal{S}$ such that $\mathrm{supp}\, \hat{\chi} \subset \{|\xi| \leq 1\}$. Let $d \geq 1$, $s \geq 0$, $0 < q \leq \infty$, and $0 < p < 1$. If $\sigma \in W(\mathcal{F}L^p, \ell^\infty)(\mathbb{R}^d)$, i.e.,

$$\|\sigma\|_{W(\mathcal{F}L^p,\ell^\infty)} = \sup_{n\in\mathbb{Z}^d} \|\sigma \cdot T_{\beta n}\chi\|_{\mathcal{F}L^p} < \infty$$

for $\beta > 0$, then H_σ extends to a bounded operator on $\mathcal{M}^{p,q}_{0,s}(\mathbb{R}^d)$.

Proof. We use the definition of the modulation spaces given by (6) (see also [9]). In particular, let $\chi \in \mathcal{S}$ such that $\mathrm{supp}\, \hat{\chi} \subset \{|\xi| \leq 1\}$, and define $g \in \mathcal{S}$ by $\hat{g} = \hat{\chi}^2$. Denote $\tilde{g}(x) = \overline{g(-x)}$. For $f \in \mathcal{S}$, $\beta > 0$, $k \in \mathbb{Z}^d$ and $x \in \mathbb{R}^d$ we have:

$$\begin{aligned}
|H_\sigma f * (M_{\beta k}\tilde{g})(x)| &= |V_g H_\sigma f(x, \beta k)| \\
&= |\langle \sigma\hat{f}, M_{-x}T_{\beta k}\hat{g}\rangle| \\
&= |\langle \sigma\hat{f}, M_{-x}T_{\beta k}\hat{\chi}^2\rangle| \\
&\leq |\mathcal{F}^{-1}(\sigma \cdot T_{\beta k}\overline{\hat{\chi}})| * |\mathcal{F}^{-1}(\hat{f} \cdot T_{\beta k}\overline{\hat{\chi}})|(x) \\
&\leq |\mathcal{F}^{-1}(\sigma \cdot T_{\beta k}\overline{\hat{\chi}})| * |f * (M_{\beta k}\tilde{\chi})|(x).
\end{aligned}$$

Now, observe that $\mathrm{supp}\big(\sigma \cdot T_{\beta k}\overline{\hat{\chi}}\big) \subset \Gamma_k := \beta k + \{|\xi| \leq 1\}$ and $\mathrm{supp}\big(\hat{f} \cdot T_{\beta k}\overline{\hat{\chi}}\big) \subset \Gamma_k$. Moreover, by assumption we know that $\sigma \in W(\mathcal{F}L^p, \ell^\infty)$ and so $\mathcal{F}^{-1}(\sigma \cdot T_{\beta k}\overline{\hat{\chi}}) \in L^p$ and $f * (M_{\beta k}\tilde{\chi}) \in L^p$. Consequently, by [9, Lemma 2.6] we have the following estimate

$$\|H_\sigma f * (M_{\beta k}\tilde{g})\|_{L^p} \leq C\, \|\mathcal{F}^{-1}(\sigma \cdot T_{\beta k}\overline{\hat{\chi}})\|_{L^p} \|f * (M_{\beta k}\tilde{\chi})\|_{L^p},$$

where C is a positive constant that depends only on the diameter of Γ_k and p. Clearly, the diameter of Γ_k is independent of k, and this makes C a constant depending only on the dimension d and the exponent p. Therefore, for $0 < q \leq \infty$ we have

$$\|H_\sigma f\|_{\mathcal{M}^{p,q}_{0,s}} \lesssim \sup_{k\in\mathbb{Z}^d} \|\mathcal{F}^{-1}(\sigma \cdot T_{\beta k}\overline{\hat{\chi}})\|_{L^p}\, \|f\|_{\mathcal{M}^{p,q}_{0,s}} = \|\sigma\|_{W(\mathcal{F}L^p,\ell^\infty)}\, \|f\|_{\mathcal{M}^{p,q}_{0,s}}.$$

The result then follows from the density of $\mathcal{S}$ in $\mathcal{M}^{p,q}_{0,s}$ for $p, q < \infty$; see [9, Theorem 3.10]. $\square$

We are now ready to state and prove the boundedness of Fourier multipliers that will be needed in establishing our main results.

Lemma 2. *Let $d \geq 1$, $s \geq 0$, and $0 < q \leq \infty$ be given. Define $m_\alpha(\xi) = e^{i|\xi|^\alpha}$. If $1 \leq p \leq \infty$ and $\alpha \in [0, 2]$, then the Fourier multiplier operator H_{m_α} extends to a bounded operator on $\mathcal{M}^{p,q}_{0,s}(\mathbb{R}^d)$.*

Moreover, If $\alpha \in \{1, 2\}$ and $\frac{d}{d+1} < p \leq \infty$, then the Fourier multiplier operator H_{m_α} extends to a bounded operator on $\mathcal{M}^{p,q}_{0,s}(\mathbb{R}^d)$.

Proof. First, we prove the result when $1 \leq p \leq \infty$, and $0 < q \leq \infty$. Let $g \in \mathcal{S}(\mathbb{R}^d)$ and define $\chi \in \mathcal{S}$ by $\hat{\chi} = g^2$. For $f \in \mathcal{S}$, we have

$$
\begin{aligned}
&|V_\chi H_{m_\alpha} f(x, \xi)| \\
&\quad = \left| \int_{\mathbb{R}^d} m_\alpha(t)\, \hat{f}(t)\, e^{2\pi i x \cdot t}\, \overline{\hat{\chi}(t - \xi)}\, dt \right| \\
&\quad = \tfrac{1}{<\xi>^s} \left| \int_{\mathbb{R}^d} m_\alpha(t)\, \overline{T_\xi g(t)}\ < t >^s\ \hat{f}(t)\, \tfrac{<\xi>^s}{<t>^s <t-\xi>^N}\ < t - \xi >^N\ \overline{g(t - \xi)}\, e^{2\pi i x \cdot t}\, dt \right| \\
&\quad = \tfrac{1}{<\xi>^s} \left| \int_{\mathbb{R}^d} m_\alpha(t)\, \overline{T_\xi g(t)}\, \phi_N(\xi, t) \widehat{< D >^s f}(t)\, T_\xi \overline{g_N(t)}\, dt \right| \\
&\quad = \tfrac{1}{<\xi>^s} \left| \mathcal{F}\Big(m_\alpha \cdot T_\xi g\, \phi_N(\xi, \cdot) \widehat{< D >^s f}\, T_\xi \overline{g_N} \Big)(-x) \right| \\
&\quad = \tfrac{1}{<\xi>^s} \left| \mathcal{F}(m_\alpha \cdot T_\xi g) * \mathcal{F}_2(\phi_N(\xi, \cdot)) * \mathcal{F}(\widehat{< D >^s f} \cdot T_\xi \overline{g_N})(-x) \right|,
\end{aligned}
$$

where $N > 0$ is an integer to be chosen later, $g_N(t) =< t >^N \overline{g}(t)$, $\phi_N(\xi, t) = \frac{<\xi>^s}{<t>^s <t-\xi>^N}$, and $< D >^s$ is the Fourier multiplier defined by $\widehat{< D >^s f}(\xi) =< \xi >^s \hat{f}(\xi)$. We also denote by $\Phi_{2,N}(\xi, \cdot) := \mathcal{F}_2(\phi_N(\xi, \cdot))$ the Fourier transform in the second variable of $\phi_N(\xi, \cdot)$

We can therefore estimate the weighted modulation norm of $H_{m_\alpha} f$ as follows:

$$\begin{aligned}
&\|H_{m_\alpha} f\|_{\mathcal{M}^{p,q}_{0,s}}\\
&= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} |V_\chi f(x,\xi)|^p\,dx\right)^{q/p} <\xi>^{qs}\right)^{1/q}\\
&= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} \left|\mathcal{F}(m_\alpha\cdot T_\xi g) * \Phi_{2,N}(\xi,\cdot) * \mathcal{F}(\widehat{<D>^s}\, f\cdot T_\xi \overline{g_N})(-x)\right|^p dx\right)^{q/p} d\xi\right)^{1/q}\\
&\leq \left(\int_{\mathbb{R}^d} \|\mathcal{F}^{-1}(m_\alpha\cdot T_\xi g)\|^q_{L^1}\, \|\Phi_{2,N}(\xi,\cdot)\|^q_{L^1}\, \|\mathcal{F}(\widehat{<D>^s}\, f\cdot T_\xi \overline{g_N})\|^q_{L^p}\, d\xi\right)^{1/q}\\
&\leq \sup_{\xi\in\mathbb{R}^d} \|\mathcal{F}^{-1}(m_\alpha\cdot T_\xi g)\|_{L^1} \sup_{\xi\in\mathbb{R}^d} \|\Phi_{2,N}(\xi,\cdot)\|_{L^1} \left(\int_{\mathbb{R}^d} \|\mathcal{F}^{-1}(\widehat{<D>^s}\, f\cdot T_\xi \overline{g_N})\|^q_{L^p}\, d\xi\right)^{1/q}
\end{aligned}$$

(8)

$$\leq \sup_{\xi\in\mathbb{R}^d} \|\mathcal{F}(m_\alpha\cdot T_\xi g)\|_{L^1} \sup_{\xi\in\mathbb{R}^d} \|\Phi_{2,N}(\xi,\cdot)\|_{L^1}\, \|f\|_{\mathcal{M}^{p,q}_{0,s}}.$$

Now, it follows from [3, Lemma 8] that, for $\alpha\in[0,2]$,

$$\sup_{\xi\in\mathbb{R}^d} \|\mathcal{F}^{-1}(m_\alpha\cdot T_\xi g)\|_{L^1} := \|m_\alpha\|_{W(\mathcal{F}L^1,\ell^\infty)} < \infty.$$

Moreover (see, for example, [13, Lemma 3.1] or [14, Lemma 2.1]), we can select a sufficiently large $N>0$ such that

$$\sup_{\xi\in\mathbb{R}^d} \|\Phi_{2,N}(\xi,\cdot)\|_{L^1} \leq \int_{\mathbb{R}^d} \sup_{\xi\in\mathbb{R}^d} |\Phi_{2,N}(\xi,x))|dx < \infty.$$

Hence, using (8), we get

$$\|H_{m_\alpha} f\|_{\mathcal{M}^{p,q}_{0,s}} \leq C_\alpha \|f\|_{\mathcal{M}^{p,q}_{0,s}}.$$

To prove the second part of the result we shall use Lemma 1. In particular, we need to show that for $\alpha\in\{1,2\}$ and $\frac{d}{d+1}<p<1$, $m_\alpha\in W(\mathcal{F}L^p,\ell^\infty)$. This, however, follows by straightforward adaptations of the proofs of [3, Theorems 9 and 11], which we leave to the interested reader. □

In analogy to the proof of the previous lemma, we can prove the following weighted version of [3, Theorem 16].

Lemma 3. *Let $d\geq 1$, $s\geq 0$, $\frac{d}{d+1}<p\leq\infty$ and $0<q\leq\infty$ be given, and let $m^{(1)}(\xi)=\frac{\sin(|\xi|)}{|\xi|}$ and $m^{(2)}(\xi)=\cos(|\xi|)$, for $\xi\in\mathbb{R}^d$. Then, the Fourier multiplier operators $H_{m^{(1)}}, H_{m^{(2)}}$ can be extended as bounded operators on $\mathcal{M}^{p,q}_{0,s}$.*

A "smooth" version of Lemma 3 is obtained by replacing $|\xi|$ with $<\xi>$.

Lemma 4. *Let $d\geq 1$, $s\geq 0$, $\frac{d}{d+1}<p\leq\infty$ and $0<q\leq\infty$ be given, and let $m(\xi)=e^{i<\xi>}$, $m^{(1)}(\xi)=\frac{\sin(<\xi>)}{<\xi>}$ and $m^{(2)}(\xi)=\cos(<\xi>)$, for $\xi\in\mathbb{R}^d$. Then, the Fourier multiplier operators $H_m, H_{m^{(1)}}, H_{m^{(2)}}$ can be extended as bounded operators on $\mathcal{M}^{p,q}_{0,s}$.*

Proof. It is clear that $m, m^{(1)}, m^{(2)}$ are $\mathcal{C}^\infty(\mathbb{R}^d)$ functions and that all their derivatives are bounded. Therefore, $m, m^{(1)}, m^{(2)} \in \mathcal{C}^{d+1}(\mathbb{R}^d) \subset \mathcal{M}^{\infty,1}(\mathbb{R}^d) \subset W(\mathcal{F}L^1, \ell^\infty)(\mathbb{R}^d)$ [8, 11]. Thus, for $1 \leq p \leq \infty$, and $0 < q \leq \infty$ the result follows from [3] and Lemma 2. For $\frac{d}{d+1} < p < 1$ and $0 < q \leq \infty$, it can be showed that $m, m^{(1)}, m^{(2)} \in \mathcal{C}^{d+1}(\mathbb{R}^d) \subset W(\mathcal{F}L^p, \ell^\infty)(\mathbb{R}^d)$. Indeed, this follows from obvious modifications to the proof of the embedding $\mathcal{C}^{d+1}(\mathbb{R}^d) \subset \mathcal{M}^{\infty,1}(\mathbb{R}^d) \subset W(\mathcal{F}L^1, \ell^\infty)(\mathbb{R}^d)$ [8, 11]. Furthermore, if we modify, for example, the multiplier m to $m_t(\xi) = e^{it<\xi>}$, $t \in \mathbb{R}$, we have for $\frac{d}{d+1} < p \leq 1$

$$\|m_t\|_{W(\mathcal{F}L^p, \ell^\infty)} \leq (1 + |t|)^{d+1}, \tag{9}$$

and similar estimates hold for modified multipliers $m_t^{(1)}$ and $m_t^{(2)}$. $\square$

Finally, we state a crucial multilinear estimate that will be used in our proofs. Although the estimate will be needed only in the particular case of a product of functions (see Corollary 1), we present it here in its full generality that applies to multilinear pseudodifferential operators.

An m-linear pseudodifferential operator is defined à priori through its (distributional) symbol σ to be the mapping T_σ from the m-fold product of Schwartz spaces $\mathcal{S} \times \cdots \times \mathcal{S}$ into the space $\mathcal{S}'$ of tempered distributions given by the formula

$$\begin{aligned} &T_\sigma(u_1, \ldots, u_m)(x) \\ &\quad = \int_{\mathbb{R}^{dm}} \sigma(x, \xi_1, \ldots, \xi_m)\, \hat{u}_1(\xi_1) \cdots \hat{u}_m(\xi_m)\, e^{2\pi i x \cdot (\xi_1 + \cdots + \xi_m)}\, d\xi_1 \cdots d\xi_m, \end{aligned} \tag{10}$$

for $u_1, \ldots, u_m \in \mathcal{S}$. The pointwise product $u_1 \cdots u_m$ corresponds to the case $\sigma = 1$.

Lemma 5. *If $\sigma \in \mathcal{M}^{\infty,1}_{0,s}(\mathbb{R}^{(m+1)d})$, then the m-linear pseudodifferential operator T_σ defined by* (10) *extends to a bounded operator from $\mathcal{M}^{p_1,q_1}_{0,s} \times \cdots \times \mathcal{M}^{p_m,q_m}_{0,s}$ into $\mathcal{M}^{p_0,q_0}_{0,s}$ when $\frac{1}{p_1} + \cdots + \frac{1}{p_m} = \frac{1}{p_0}$, $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = m - 1 + \frac{1}{q_0}$, and $0 < p_i \leq \infty, 1 \leq q_i \leq \infty$ for $0 \leq i \leq m$.*

This result is a slight modification of [4, Theorem 3.1]. Its proof proceeds along the same lines, and therefore it is omitted here. Note that if $\sigma \in \mathcal{M}^{\infty,1}_{0,s}$, and we pick $u_1 = \cdots = u_m = u$ (some of them could be equal to $\bar{u}$ since the modulation norm is preserved), $p_1 = \cdots = p_m = mp$, $0 < p \leq \infty$, and $q_1 = \cdots = q_m = 1$ we have

$$\|T_\sigma(u, \ldots, u)\|_{\mathcal{M}^{p,1}_{0,s}} \lesssim \|u\|^m_{\mathcal{M}^{mp,1}_{0,s}} \lesssim \|u\|^m_{\mathcal{M}^{p,1}_{0,s}}, \tag{11}$$

where we used the obvious embedding $\mathcal{M}^{p,1}_{0,s} \subseteq \mathcal{M}^{mp,1}_{0,s}$. The notation $A \lesssim B$ stands for $A \leq cB$ for some positive constant c independent of A and B. In particular, if we select $\sigma = 1$ (the constant function 1), then $\sigma \in \mathcal{M}^{\infty,1}_{0,s} \subset \mathcal{M}^{\infty,1}$, and we obtain

Corollary 1. *Let $0 < p \leq \infty$. If $u \in \mathcal{M}^{p,1}_{0,s}$, then $u^m \in \mathcal{M}^{p,1}_{0,s}$. Furthermore,*

$$\|u^m\|_{\mathcal{M}^{p,1}_{0,s}} \lesssim \|u\|^m_{\mathcal{M}^{p,1}_{0,s}}.$$

This is of course just a particular case of the more general multilinear estimate

$$\Big\| \prod_{i=1}^m u_i \Big\|_{\mathcal{M}_{0,s}^{p_0,q_0}} \lesssim \prod_{i=1}^m \|u_i\|_{\mathcal{M}_{0,s}^{p_i,q_i}}, \tag{12}$$

where the exponents satisfy the same relations as in Lemma 1. When we consider the power nonlinearity $f(u) = p_k(u) = \lambda |u|^{2k} u = \lambda u^{k+1}\bar{u}^k$, Corollary 1 becomes

Corollary 2. *Let* $0 < p \le \infty$. *If* $u \in \mathcal{M}_{0,s}^{p,1}$, *then* $p_k(u) \in \mathcal{M}_{0,s}^{p,1}$. *Furthermore,*

$$\|p_k(u)\|_{\mathcal{M}_{0,s}^{p,1}} \lesssim \|u\|_{\mathcal{M}_{0,s}^{p,1}}^{2k+1}.$$

For a different proof of the estimate in Corollary 2, see [1, Corollary 4.2]. It is important to note that the previous estimate allows us to control the exponential nonlinearity e_ρ as well. Indeed, since

$$e_\rho(u) = \lambda(e^{\rho|u|^2} - 1)u = \sum_{k=1}^\infty \tfrac{\rho^k}{k!} p_k(u),$$

if we now apply the modulation norm on both sides and use the triangle inequality, we arrive at

Corollary 3. *Let* $0 < p \le \infty$. *If* $u \in \mathcal{M}_{0,s}^{p,1}$, *then* $e_\rho(u) \in \mathcal{M}_{0,s}^{p,1}$. *Furthermore,*

$$\|e_\rho(u)\|_{\mathcal{M}_{0,s}^{p,1}} \lesssim \|u\|_{\mathcal{M}_{0,s}^{p,1}} (e^{|\rho| \|u\|^2_{\mathcal{M}_{0,s}^{p,1}}} - 1).$$

3. Proofs of the main results

We are now ready to proceed with the proofs of our main theorems. We will only prove our results for the power nonlinearities $f = p_k$, by making use of Corollary 2. The case of exponential nonlinearity $f = e_\rho$ is treated similarly, by now employing Corollary 3. In all that follows we assume that $u : [0, T) \times \mathbb{R}^d \to \mathbb{C}$ where $0 < T \le \infty$ and that $f(u) = p_k(u) = \lambda |u|^{2k} u$.

3.1. The nonlinear Schrödinger equation: Proof of Theorem 1.

We start by noting that (1) can be written in the equivalent form

$$u(\cdot, t) = S(t)u_0 - i\mathcal{A} f(u) \tag{13}$$

where

$$S(t) = e^{it\Delta}, \quad \mathcal{A} = \int_0^t S(t-\tau) \cdot \, d\tau. \tag{14}$$

Consider now the mapping

$$\mathcal{J} u = S(t)u_0 - i \int_0^t S(t-\tau)(p_k(u))(\tau)\, d\tau.$$

It follows from Lemma 2 (see also [3, Corollary 18]) that

$$\|S(t)u_0\|_{\mathcal{M}_{0,s}^{p,1}} \le C\,(t^2 + 4\pi^2)^{d/4}\, \|u_0\|_{\mathcal{M}_{0,s}^{p,1}},$$

where C is a universal constant depending only on d. Therefore,

$$\|S(t)u_0\|_{\mathcal{M}^{p,1}_{0,s}} \leq C_T \|u_0\|_{\mathcal{M}^{p,1}_{0,s}}, \tag{15}$$

where $C_T = \sup_{t\in[0,T)} C\,(t^2+4\pi^2)^{d/4}$. Moreover, we have

$$\begin{aligned}\left\| \int_0^t S(t-\tau)(p_k(u))(\tau)\,d\tau \right\|_{\mathcal{M}^{p,1}_{0,s}} &\leq \int_0^t \|S(t-\tau)(p_k(u))(\tau)\|_{\mathcal{M}^{p,1}_{0,s}}\,d\tau \\ &\leq T\,C_T \sup_{t\in[0,T]} \|p_k(u)(t)\|_{\mathcal{M}^{p,1}_{0,s}}.\end{aligned} \tag{16}$$

By using now Corollary 2, we can further estimate in (16) to get

$$\left\| \int_0^t S(t-\tau)(p_k(u))(\tau)\,d\tau \right\|_{\mathcal{M}^{p,1}_{0,s}} \lesssim C_T\,T\,\|u(t)\|^{2k+1}_{\mathcal{M}^{p,1}_{0,s}}. \tag{17}$$

Consequently, using (15) and (17) we have

$$\|\mathcal{J}u\|_{C([0,T],\mathcal{M}^{p,1}_{0,s})} \leq C_T(\|u_0\|_{\mathcal{M}^{p,1}_{0,s}} + cT\,\|u\|^{2k+1}_{\mathcal{M}^{p,1}_{0,s}}), \tag{18}$$

for some universal positive constant c. We are now in the position of using a standard contraction argument to arrive to our result. For completeness, we sketch it here. Let $\mathbf{B}_M$ denote the closed ball of radius M centered at the origin in the space $C([0,T],\mathcal{M}^{p,1}_{0,s})$. We claim that

$$\mathcal{J} : \mathbf{B}_M \to \mathbf{B}_M,$$

for a carefully chosen M. Indeed, if we let $M = 2C_T\|u_0\|_{\mathcal{M}^{p,1}_{0,s}}$ and $u \in \mathbf{B}_M$, from (18) we obtain

$$\|\mathcal{J}u\|_{C([0,T],\mathcal{M}^{p,1}_{0,s})} \leq \frac{M}{2} + cC_T T M^{2k+1}.$$

Now let T be such that $cC_T T M^{2k} \leq 1/2$, that is, $T \leq \tilde{T}(\|u_0\|_{\mathcal{M}^{p,1}_{0,s}})$. We obtain

$$\|\mathcal{J}u\|_{C([0,T],\mathcal{M}^{p,1}_{0,s})} \leq \frac{M}{2} + \frac{M}{2} = M,$$

that is $\mathcal{J}u \in \mathbf{B}_M$. Furthermore, a similar argument gives

$$\|\mathcal{J}u - \mathcal{J}v\|_{C([0,T],\mathcal{M}^{p,1}_{0,s})} \leq \frac{1}{2}\|u-v\|_{C([0,T],\mathcal{M}^{p,1}_{0,s})}.$$

This last estimate follows in particular from the following fact:

$$p_k(u)(\tau) - p_k(v)(\tau) = \lambda(u-v)|u|^{2k}(\tau) + \lambda v(|u|^{2k} - |v|^{2k})(\tau).$$

Therefore, using Banach's contraction mapping principle, we conclude that $\mathcal{J}$ has a fixed point in $\mathbf{B}_M$ which is a solution of (13); this solution can be now extended up to a maximal time $T^*(\|u_0\|_{\mathcal{M}^{p,1}_{0,s}})$. The proof is complete.

3.2. **The nonlinear wave equation: Proof of Theorem 2.** Equation (2) can be written in the equivalent form

$$u(\cdot,t) = \tilde{K}(t)u_0 + K(t)u_1 - \mathcal{B}f(u) \tag{19}$$

where

$$K(t) = \tfrac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}},\ \tilde{K}(t) = \cos(t\sqrt{-\Delta}),\ \mathcal{B} = \int_0^t K(t-\tau)\cdot d\tau \tag{20}$$

Consider the mapping

$$\mathcal{J}u = \tilde{K}(t)u_0 + K(t)u_1 - \mathcal{B}f(u).$$

Recall that $f = p_k$. If we now use Lemma 3 (see also [3, Corollary 21]) for the first two inequalities below and Corollary 2 for the last estimate, we can write

$$\begin{cases} \|\tilde{K}(t)u_0\|_{\mathcal{M}^{p,1}_{0,s}} \leq C_T\|u_0\|_{\mathcal{M}^{p,1}_{0,s}}, \\ \|K(t)u_1\|_{\mathcal{M}^{p,1}_{0,s}} \leq C_T\|u_1\|_{\mathcal{M}^{p,1}_{0,s}}, \\ \|\mathcal{B}f(u)\|_{\mathcal{M}^{p,1}_{0,s}} \leq cT\,C_T\|u\|^{2k+1}_{\mathcal{M}^{p,1}_{0,s}}, \end{cases} \tag{21}$$

where c is some universal positive constant. The constants T and C_T have the same meaning as before. The standard contraction mapping argument applied to $\mathcal{J}$ completes the proof.

3.3. **The nonlinear Klein-Gordon equation: Proof of Theorem 3.** The equivalent form of equation (3) is

$$u(\cdot,t) = \tilde{K}(t)u_0 + K(t)u_1 + \mathcal{C}f(u) \tag{22}$$

where now

$$K(t) = \tfrac{\sin t(I-\Delta)^{1/2}}{(I-\Delta)^{1/2}},\ \tilde{K}(t) = \cos t(I-\Delta)^{1/2},\ \mathcal{C} = \int_0^t K(t-\tau)\cdot d\tau. \tag{23}$$

Consider the mapping

$$\mathcal{J}u = \tilde{K}(t)u_0 + K(t)u_1 + \mathcal{C}f(u).$$

Using Lemma 4 and the notations above, we can write

$$\begin{cases} \|\tilde{K}(t)u_0\|_{\mathcal{M}^{p,1}_{0,s}} \leq C_T\|u_0\|_{\mathcal{M}^{p,1}_{0,s}}, \\ \|K(t)u_1\|_{\mathcal{M}^{p,1}_{0,s}} \leq C_T\|u_1\|_{\mathcal{M}^{p,1}_{0,s}}, \\ \|\mathcal{C}f(u)\|_{\mathcal{M}^{p,1}_{0,s}} \leq cT\,C_T\|u\|^{2k+1}_{\mathcal{M}^{p,1}_{0,s}}, \end{cases} \tag{24}$$

The standard contraction mapping argument applied to $\mathcal{J}$ completes the proof.

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